数理解析研究所講究録 第2125巻 2019年 52-60

On Spaces of Lipschitz Maps with Values in a Uniform Algebra

Kyohei Tanabe *
Master's Program in Fundamental Sciences
Graduate School of Science and Technology
Niigata University

In this paper, linear always means complex linear, especially Banach algebra always means complex Banach algebra. Isometries with respect to the s-norm between vector valued Lipschitz spaces were studied by Hatori and Oi [2]. We prove a version of their results (Main Theorem A). There are literatures which study isometries with respect to the max-norm between vector valued Lipschitz spaces [1, 4]. In this paper, we exhibit the form of isometries with respect to the max-norm under an additional condition (Main Theorem B) (cf. [7].).

1 Definitions

In this section, we introduce some basic definitions.

Definition 1.1. Let X be a compact metric space and E a normed space. A map $f : X \longrightarrow E$ is called *a Lipschitz map* if

$$(L(f) :=) \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x) - f(y)\|_E}{d(x,y)} < \infty.$$

The number $L(f)$ is called *the Lipschitz constant of* f. We shall denote by $\mathrm{Lip}(X,E)$ the space of all Lipschitz maps from X into E. We write the space $\mathrm{Lip}(X,\mathbb{C})$ just by

* This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

Lip(X) for simplification. There are several norms on the Lipschitz space Lip(X, E): *s-norm* $\|\cdot\|_s$ is defined by $\|\cdot\|_s = \|\cdot\|_\infty + L(\cdot)$, and *max-norm* $\|\cdot\|_{max}$ by $\|\cdot\|_{max} = \max\{\|\cdot\|_\infty, L(\cdot)\}$. If B is a Banach algebra, the space $(\mathrm{Lip}(X, B), \|\cdot\|_s)$ is a Banach algebra, and the space $(\mathrm{Lip}(X, B), \|\cdot\|_{max})$ is a Banach space (In general, submultiplicativity needs not hold.).

2 Main Theorem A

The next is the theorem of isometries with respect to the s-norm between Lipschitz spaces. In this section, we give an outline of the proof of this theorem.

Theorem 2.1 (Main Theorem A)**.** *For $j = 1, 2$, let X_j be a compact metric space, Y_j a compact Hausdorff space, and A_j a uniform algebra. If $U : (\mathrm{Lip}(X_1, A_1), \|\cdot\|_s) \longrightarrow (\mathrm{Lip}(X_2, A_2), \|\cdot\|_s)$ is a unital surjective linear isometry, then there exist*

- *a continuous map $\psi : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$ such that for every $y' \in \mathrm{Ch}(A_2)$ $\psi(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry,*

and

- *a homeomorphism $\tau : \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$*

such that $(U(F)(x'))(y') = (F(\psi(x', y')))(\tau(y'))$ for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, and $F \in \mathrm{Lip}(X_2, A_2)$.

Remark 2.2. A map U being unital means $U(1) = 1$. The space $\mathrm{Ch}(A)$ denotes a Choquet boundary of A. (If Y is a compact metric space and A is a subspace of $(C(Y), \|\cdot\|_\infty)$, $\mathrm{Ch}(A) = \{y \in Y \mid \tau_y \in \mathrm{ext}\{\varphi \in A^* \mid \|\varphi\| = \varphi(1) = 1\}\}$ where τ_y is the evaluation map at $y \in Y$.)

Outline of the proof of the Main Theorem A

First, we regard $\mathrm{Lip}(X_j, A_j)$ as a subspace of $C(X_j \times Y_j)$. We apply a theorem of Jarosz [3], then we find that U is an isometry also with respect to the supremum norm. Using partition of unity, we find that the uniform closure of $\mathrm{Lip}(X_j, A_j)$ coincides with $C(X_j, A_j)$. So we can extend U from $C(X_1, A_1)$ onto $C(X_2, A_2)$ which is a unital surjective linear isometry with respect to the supremum norm. We denote this map by $\tilde{U}^{\infty}$. We define maps

$$\begin{aligned} S : &\left\{\varphi' \in C(X_2, A_2)^* \mid \|\varphi'\| = \varphi'(1) = 1\right\} \\ &\longrightarrow \left\{\varphi \in C(X_1, A_1)^* \mid \|\varphi\| = \varphi(1) = 1\right\} \end{aligned}$$

by $S(\varphi') := \varphi' \circ \tilde{U}^{\infty}$ and

$$\begin{aligned} S' : &\left\{\varphi \in C(X_1, A_1)^* \mid \|\varphi\| = \varphi(1) = 1\right\} \\ &\longrightarrow \left\{\varphi' \in C(X_2, A_2)^* \mid \|\varphi'\| = \varphi'(1) = 1\right\} \end{aligned}$$

by $S'(\varphi) := \varphi \circ \left(\tilde{U}^{\infty}\right)^{-1}$. Then, S and S' are well-defined, S' is an inverse map of S, and S is a w^*-homeomorphism.
For $j = 1, 2$, we define a set

$$K_j := \operatorname{ext}\left\{\varphi \in C(X_j, A_j)^* \mid \|\varphi\| = \varphi(1) = 1\right\}.$$

Then we find that $S(K_2) = K_1$ by some easy argument of extreme points. We note that the Choquet boundary of $C(X_j, A_j)$ coincides with $X_j \times \mathrm{Ch}(A_j)$. If we define a homeomorphism $\Phi_j : X_j \times \mathrm{Ch}(A_j) \longrightarrow K_j$ by $\Phi_j(x, y) = \varphi_{(x,y)}$ where $\varphi_{(x,y)}$ is the evaluation at (x, y) for $j = 1, 2$, then the map $\Phi_1^{-1} \circ S \circ \Phi_2$ is a homeomorphism between $X_2 \times \mathrm{Ch}(A_2)$ and $X_1 \times \mathrm{Ch}(A_1)$. So we can define continuous maps $\psi_1 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$, $\psi_2 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$ by $(\psi_1, \psi_2) = \Phi_1^{-1} \circ S \circ \Phi_2$. By the similar way, we consider the homeomorphism $\Phi_2^{-1} \circ S^{-1} \circ \Phi_1$ between $X_1 \times \mathrm{Ch}(A_1)$ and $X_2 \times \mathrm{Ch}(A_2)$, and define continuous maps $\psi_1' : X_1 \times \mathrm{Ch}(A_1) \longrightarrow X_2$, $\psi_2' : X_1 \times \mathrm{Ch}(A_1) \longrightarrow \mathrm{Ch}(A_2)$ by $(\psi_1', \psi_2') = \Phi_2^{-1} \circ S^{-1} \circ \Phi_1$. Then for every $x' \in X_2$ and $y' \in \mathrm{Ch}(A_2)$, $((U(F))(x'))(y') = S\left(\varphi'_{(x',y')}\right)(F) = (F(\psi_1(x', y')))(\psi_2(x', y'))$. We shall observe the maps ψ_1, ψ_2.

At the first, We show that the map ψ_2 needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2(x_1', y') = \psi_2(x_2', y')$ holds for any $x_1', x_2' \in X_2$ and $y' \in \mathrm{Ch}(A_2)$. Suppose that there are $x_1^\circ \neq x_2^\circ \in X_2$ and $y^\circ \in \mathrm{Ch}(A_2)$ such that $\psi_2(x_1^\circ, y^\circ) \neq \psi_2(x_2^\circ, y^\circ)$. Then there is $h \in A_1$ such that $h(\psi_2(x_1^\circ, y^\circ)) \neq h(\psi_2(x_2^\circ, y^\circ))$ since A_1 is a uniform algebra. By the direct computation, we assert that $L(1 \otimes h) = 0$ and $L(U(1 \otimes h)) \neq 0$. On the other hand U preserves the Lipschitz constant because U preserves the s-norm and the supremum norm. This is a contradiction. Hence the map ψ_2 needs not depend on the first variable.

Then we define continuous maps $\tau : \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$ by $\tau(y') = \psi_2(x', y')$ ($y' \in \mathrm{Ch}(A_2)$) for some $x' \in X_2$, and $\tau' : \mathrm{Ch}(A_1) \longrightarrow \mathrm{Ch}(A_2)$ by $\tau'(y) = \psi_2'(x, y)$

$(y \in \mathrm{Ch}(A_1))$ for some $x \in X_1$. We can check that the map τ is a homeomorphism between $\mathrm{Ch}(A_2)$ and $\mathrm{Ch}(A_1)$. Moreover, τ' is an inverse map of τ. On the other hand, the maps $\psi_1(\cdot, y') : X_2 \longrightarrow X_1$ and $\psi_1' : X_1 \longrightarrow X_2$ are bijective for each $y' \in \mathrm{Ch}(A_2)$ and $y \in \mathrm{Ch}(A_1)$ respectively. Moreover, $\psi_1(\cdot, y') = \psi_1'(\cdot, \tau(y'))^{-1}$ and $\psi_1'\left(\cdot, \tau^{-1}(y)\right)^{-1}$ hold for each $y' \in \mathrm{Ch}(A_2)$ and $y \in \mathrm{Ch}(A_1)$ respectively.

These indicate that it is sufficient to show that $\psi_1(\cdot, y_0') : X_2 \longrightarrow X_1$ is a contractive map for each $y_0' \in \mathrm{Ch}(A_2)$ which proves the Main Theorem A. Take $y_0' \in \mathrm{Ch}(A_2)$ arbitrarily. We prove that $d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) \leq d(x_1', x_2')$ for every distinct $x_1', x_2' \in X_2$. We define a function $f_{\psi_1(x_2', y_0')} : X_1 \longrightarrow \mathbb{C}$ by $f_{\psi_1(x_2', y_0')}(x) = d(x, \psi_1(x_2', y_0'))$ for $x \in X_1$. Then $f_{\psi_1(x_2', y_0')}$ is in $\mathrm{Lip}(X_1)$ and $L\left(f_{\psi_1(x_2', y_0')}\right) = 1$. Therefore,

$$\begin{aligned}
& d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) \\
&= d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) - d(\psi_1(x_2', y_0'), \psi_1(x_2', y_0')) \\
&= \left| f_{\psi_1(x_2', y_0')}(\psi_1(x_1', y_0')) - f_{\psi_1(x_2', y_0')}(\psi_1(x_2', y_0')) \right| \\
&= \left| \left(\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right)(\psi_1(x_1', y_0')) \right)(\tau(y_0')) - \left(\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right)(\psi_1(x_2', y_0')) \right)(\tau(y_0')) \right| \\
&= \left| \left(\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_1') \right)(y_0') - \left(\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_2') \right)(y_0') \right| \\
&\leq \left\| \left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_1') - \left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right)(x_2') \right\|_{\infty(Y_2)} \\
&\leq d(x_1', x_2')\, L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right).
\end{aligned}$$

Since U preserves the Lipschitz constant, we have

$$L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right) = L\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) = L\left(f_{\psi_1(x_2', y_0')} \right) = 1.$$

Hence we have

$$\begin{aligned}
d(\psi_1(x_1', y_0'), \psi_1(x_2', y_0')) &\leq d(x_1', x_2')\, L\left(U\left(f_{\psi_1(x_2', y_0')} \otimes 1 \right) \right) \\
&= d(x_1', x_2')
\end{aligned}$$

and $\psi_1(\cdot, y_0')$ is a contractive map. We complete the outline of the proof of the Main Theorem A.

$\square$

3 Main Theorem B

In this section, we consider isometries with respect to the max-norm between Lipschitz spaces. We exhibit the Main Theorem B and give an outline of the proof of this theorem.

Definition 3.1 (K pair)**.** Let X_1 and X_2 be compact metric spaces. We say that the ordered pair (X_1, X_2) of these two sets is *a K pair* if the following two conditions are satisfied.

- (K 1) For $j = 1, 2$, if we take any $x_1, x_2 \in X_j$, there are finitely many $x_1^\circ, ..., x_n^\circ \in X_j$ such that $d(x_1, x_1^\circ) < 1$, $d\left(x_i^\circ, x_{i+1}^\circ\right) < 1$ $(i = 1, ...n-1)$, $d(x_n^\circ, x_2) < 1$.
- (K 2) For any bijection $\psi : X_2 \longrightarrow X_1$ and positive ε, the following statement holds; if $x_1', x_2' \in X_2$ and $d(x_1', x_2') < \varepsilon$ implies that $d(\psi(x_1'), \psi(x_2')) = d(x_1', x_2')$, then ψ is an isometry.

Theorem 3.2 (Main Theorem B)**.** *For $j = 1, 2$, let X_j be a compact metric space, Y_j a compact Hausdorff space, and A_j a uniform algebra. We assume that (X_1, X_2) is a K pair. If $U : (\mathrm{Lip}(X_1, A_1), \|\cdot\|_{max}) \longrightarrow (\mathrm{Lip}(X_2, A_2), \|\cdot\|_{max})$ is a unital surjective linear isometry, then there exist*

- *a continuous map $\psi : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$ such that for every $y' \in \mathrm{Ch}(A_2)$ $\psi(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry,*

and

- *a homeomorphism $\tau : \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$*

such that $(U(F)(x'))(y') = (F(\psi(x', y')))(\tau(y'))$ for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, and $F \in \mathrm{Lip}(X_2, A_2)$.

Outline of the proof of the Main Theorem B

We can prove by the same way as the outline of the proof of Theorem 2.1 that there are continuous maps $\psi_1 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow X_1$, $\psi_2 : X_2 \times \mathrm{Ch}(A_2) \longrightarrow \mathrm{Ch}(A_1)$ such that for every $x' \in X_2$, $y' \in \mathrm{Ch}(A_2)$, $((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))$ holds.

At the first, we show that the map ψ_2 needs not depend on the first variable $x' \in X_2$, that is, the equality $\psi_2(x'_1, y') = \psi_2(x'_2, y')$ holds for any $x'_1, x'_2 \in X_2$ and $y' \in \mathrm{Ch}(A_2)$. By the condition (K 1), it suffices to show that the equality $\psi_2(x'_1, y') = \psi_2(x'_2, y')$ holds for every $x'_1, x'_2 \in X_2$ with $d(x'_1, x'_2) < 1$ and $y' \in \mathrm{Ch}(A_2)$. If not, there exist $x_1^\circ, x_2^\circ \in X_2$ with $d(x_1^\circ, x_2^\circ) < 1$ and $y^\circ \in \mathrm{Ch}(A_2)$ such that $\psi_2(x_1^\circ, y^\circ) \neq \psi_2(x_2^\circ, y^\circ)$. Let $\varepsilon_0 = \frac{1-d(x_1^\circ, x_2^\circ)}{2}$. We take an open neighborhood $V \subset Y_1$ of $\psi_2(x_1^\circ, y^\circ)$ which doesn't contain $\psi_2(x_2^\circ, y^\circ)$. Then there is a peaking function $h \in A_1$ such that $h(\psi_2(x_1^\circ, y^\circ)) = 1$, and $|h(y)| < \varepsilon_0$ for every $y \in Y_1 \setminus V$. Especially $|h(\psi_2(x_2^\circ, y^\circ))| < \varepsilon_0$. It is clear that $\|1 \otimes h\|_{max} = 1$. On the other hand,

$$\begin{aligned} L(U(1 \otimes h)) &\geq \frac{|U(1 \otimes h)(x_1^\circ, y^\circ) - U(1 \otimes h)(x_2^\circ, y^\circ)|}{d(x_1^\circ, x_2^\circ)} \\ &= \frac{|h(\psi_2(x_1^\circ, y^\circ)) - h(\psi_2(x_2^\circ, y^\circ))|}{d(x_1^\circ, x_2^\circ)} \\ &> \frac{1 - 2\varepsilon_0}{d(x_1^\circ, x_2^\circ)} = 1 \end{aligned}$$

holds, hence we get $\|U(1 \otimes h)\|_{max} > 1$. This contradicts to the fact that U preserves the max-norm. Thus ψ_2 needs not depend on the first variable.

By the Theorem of Jarosz [3], U is also an isometry with respect to the supremum norm. We can extend U from the uniform closure of $\mathrm{Lip}(X_1, A_1)$, which is $C(X_1, A_1)$, onto the uniform closure of $\mathrm{Lip}(X_2, A_2)$, which is $C(X_2, A_2)$, that is a unital surjective linear isometry with respect to the supremum norm. Since $C(X_j, A_j)$ is a uniform algebra, a theorem of Nagasawa [5] yields that U is multiplicative. For each $y' \in \mathrm{Ch}(A_2)$, we define a map $U_{y'} : \mathrm{Lip}(X_1) \longrightarrow \mathrm{Lip}(X_2)$ by

$$U_{y'}(f) = ((U(f \otimes 1))(\cdot))(y')$$

for $f \in \mathrm{Lip}(X_1)$. $U_{y'}$ is a unital homomorphism. So by [6, Theorem 5-1] , there is a Lipschitz map $\phi_{y'} : X_2 \longrightarrow X_1$ such that $U_{y'}(f) = f \circ \phi_{y'}$ for every $f \in \mathrm{Lip}(X_1)$. It is easy to check the equality $\phi_{y'} = \psi_1(\cdot, y')$. Hence $\psi_1(\cdot, y')$ is a Lipschitz map.

Next we prove that $\psi_1(\cdot, y') : X_2 \longrightarrow X_1$ is a surjective isometry for each $y' \in \mathrm{Ch}(A_2)$. Let $\varepsilon_0 = \frac{1}{\max\{1, L(\psi_1(\cdot, y'))\}}$. By (K 2) and the descriptions in the outline of the proof of Theorem 2.1, it suffices to show that for every $x'_1, x'_2 \in X_2$ with $d(x'_1, x'_2) < \varepsilon_0$, the equality $d(x'_1, x'_2) \geq d(\psi_1(x'_1, y'), \psi_1(x'_2, y'))$ holds. We define $f_{\psi_1(x'_2, y')} \in \mathrm{Lip}(X_1)$ by $f_{\psi_1(x'_2, y')}(x) = \min\{d(x, \psi_1(x'_2, y')), 1\}$ for

$x \in X_1$, then we have $L\left(f_{\psi_1(x_2',y')}\right) \leq 1$, $\left\|f_{\psi_1(x_2',y')}\right\|_\infty \leq 1$. By the definition of ε_0, we get $d\left(\psi_1\left(x_1',y'\right),\psi_1\left(x_2',y'\right)\right) \leq 1$. Hence $f_{\psi_1(x_2',y')}\left(\psi_1\left(x_1',y'\right)\right) = d\left(\psi_1\left(x_1',y'\right),\psi_1\left(x_2',y'\right)\right)$, and

$$\begin{aligned}
& d\left(\psi_1\left(x_1',y'\right),\psi_1\left(x_2',y'\right)\right) \\
&= \left|f_{\psi_1(x_2',y')}\left(\psi_1\left(x_1',y'\right)\right) - f_{\psi_1(x_2',y')}\left(\psi_1\left(x_2',y'\right)\right)\right| \\
&= \left|\left(U\left(f_{\psi_1(x_2',y')}\otimes 1\right)\left(x_1'\right)\right)\left(y'\right) - \left(U\left(f_{\psi_1(x_2',y')}\otimes 1\right)\left(x_2'\right)\right)\left(y'\right)\right| \\
&\leq \left\|U\left(f_{\psi_1(x_2',y')}\otimes 1\right)\left(x_1'\right) - U\left(f_{\psi_1(x_2',y')}\otimes 1\right)\left(x_2'\right)\right\|_{\infty(Y_2)} \\
&\leq L\left(U\left(f_{\psi_1(x_2',y')}\otimes 1\right)\right) d\left(x_1',x_2'\right) \leq d\left(x_1',x_2'\right).
\end{aligned}$$

Thus we get the desired inequality. Now we complete the outline of the proof of the Main Theorem B.

□

In the next section, we observe some examples of K pairs, and Main Theorem B without the condition, K pair.

4 K pairs

In the Main Theorem B, we assume that (X_1, X_2) is a K pair. We give some examples of K pairs.

Example 4.1.

1. If $a < b$, the pair of closed intervals $([a,b],[a,b])$ is a K pair.
2. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with the usual distance, then $\left(\overline{\mathbb{D}}, \overline{\mathbb{D}}\right)$ is a K pair.
3. Let $K = (\{0\} \times [-1,1]) \cup \left(\left\{\left(t, \frac{1}{2}t\right) \mid 0 \leq t \leq 2\right\}\right) \cup \left(\left\{\left(t, -\frac{1}{2}t\right) \mid 0 \leq t \leq 2\right\}\right) \subset \mathbb{R}^2$ with the usual distance, then (K, K) is a K pair.

It is not difficult to check that these three pairs above are K pairs.

Example 4.2. Let $X_1 = X_2 = Y_1 = Y_2 = \{a, b\}$ where the distance of a and b is 2, then (X_1, X_2) is not a K pair because it doesn't satisfy (K 1). We define a map $\phi : X_2 \times Y_2 \longrightarrow X_1 \times Y_1$ by

$$\phi\left((a,a)\right) = (a,a)\,,\ \phi\left((a,b)\right) = (b,a)$$

$$\phi((b,a)) = (a,b), \ \phi((b,b)) = (b,b)$$

and maps $\psi_1 : X_2 \times Y_2 \longrightarrow X_1$, $\psi_2 : X_2 \times Y_2 \longrightarrow Y_1$ by $\phi = (\psi_1, \psi_2)$. Let U : $\mathrm{Lip}(X_1, C(Y_1)) \longrightarrow \mathrm{Lip}(X_2, C(Y_2))$ be

$$((U(F))(x'))(y') = (F(\psi_1(x', y')))(\psi_2(x', y'))$$

for $x' \in X_2$, $y' \in Y_2$, and $F \in \mathrm{Lip}(X_1, C(Y_1))$. Then U is a unital surjective linear isometry with respect to the max-norm. Actually for every $F \in \mathrm{Lip}(X_j, C(Y_j))$,

$$L(F) = \frac{\|F(a) - F(b)\|_\infty}{2} \leq \|F\|_\infty .$$

Hence the max-norm coincides with the supremum norm in this case. The map U is clearly an isometry with respect to the supremum norm. But U cannot be represented as the form in Theorem 3.2.

Example 4.3. Let $H = (\{0\} \times [-1,1]) \cup ([0,3] \times \{0\}) \cup (\{3\} \times [-1,1]) \subset \mathbb{R}^2$ with the usual distance. Then (H, H) is not a K pair. To prove this, we define a bijection $\psi : H \longrightarrow H$ by

$$\psi((x,y)) = \begin{cases} (x,y) & ((x,y) \in \{0\} \times [-1,1]) \\ (x,y) & ((x,y) \in [0,3] \times \{0\}) \\ (x,-y) & ((x,y) \in \{3\} \times [-1,1]) . \end{cases}$$

Then $d((x_1,y_1),(x_2,y_2)) < 2$ implies that $d((x_1,y_1),(x_2,y_2)) = d(\psi((x_1,y_1)), \psi((x_2,y_2)))$ but ψ is not an isometry. Let Y be any compact Hausdorff space. We define a map $U : \mathrm{Lip}(H, C(Y)) \longrightarrow \mathrm{Lip}(H, C(Y))$ by

$$((U(F))(x'))(y') = (F(\psi(x')))(y')$$

for $x' \in H$, $y' \in Y$, and $F \in \mathrm{Lip}(H, C(Y))$. This U is not represented by the form in Theorem 3.2, but U is a unital surjective linear isometry with respect to the max-norm.

References

[1] Botelho, Fernanda; Fleming, Richard J.; Jamison, James E. *Extreme points and isometries on vector-valued Lipschitz spaces.* J. Math. Anal. Appl. **381** (2011), no. 2, 821-832.

[2] Hatori, Osamu; Oi, Shiho, *Hermitian operators on Banach algebras of vector-valued Lipschitz maps.* J. Math. Anal. Appl. **452** (2017), no. 1, 378-387.

[3] Jarosz, Krzysztof, *Isometries in semisimple, commutative Banach algebras.* Proc. Amer. Math. Soc. **94** (1985), no. 1, 65-71.

[4] Jiménez-Vargas, A.; Villegas-Vallecillos, Moisés, *Linear isometries between spaces of vector-valued Lipschitz functions.* Proc. Amer. Math. Soc. **137** (2009), no. 4, 1381-1388.

[5] Nagasawa, Masao, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions.* Kōdai Math. Sem. Rep. **11** (1959), 182-188. 46.00

[6] Sherbert, Donald R. *Banach algebras of Lipschitz functions.* Pacific J. Math. **13** (1963), 1387-1399.

[7] Tanabe, Kyohei, *Isometries on Spaces of Lipschitz Maps with Values in a Uniform Algebra*, master's thesis in Niigata University (2019)